A Note on Coverings by Radial Rule Bases

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Abstract

The paper discusses covering of an input space by a rule base of a radial fuzzy system. We address the question of what is the minimal degree of firing of a radial rule base across the input space? This minimal degree is called the degree of covering (DOC). For radial rule bases, a search for the DOC leads to a constrained optimization problem. This generally hard problem can be eased by passing to convex optimization, however, for the price of obtaining only a lower bound for DOC.

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maketitle

1 Introduction

The heart of a fuzzy system is its rule base. It is well known that, mathematically, it represents a fuzzy relation on the Cartesian product of input and output spaces of the system. The relation is canonically established on the basis of individual IF-THEN rules that are combined into the rule base.

A set of IF-THEN rules constituting a rule base of a fuzzy system can be identified by questioning experts in a given field. This approach is effective if the relation encoded by the rule base is not highly multidimensional and maximally dozens of rules are introduced. The more progressive way, due to the rise in technology of data acquisition and storage, is to establish the fuzzy system's rule base directly from data associated with the field of interest.

A question arises, especially in connection with the data-driven approach, whether there is for each input at least one rule that is fired in a sufficient degree, i.e., if we are ensured by a certain minimal degree of firing (DOF) for each possible input. This can be interpreted that we require that our knowledge about the domain of interest the fuzzy system operates on is sufficiently reliable.

If a fuzzy system is build up directly from data, then rules are typically created on the basis of some clustering algorithm and no regular structure is

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recognized on introduced fuzzy sets. The reason is that clustering algorithms locate antecedents of IF-THEN rules into the areas with a high density of clustered data and cases of uncovering of other areas may arise. As a consequence, a given minimal degree of firing across the input space, which we call the *degree of covering* (DOC), is not inherently guaranteed and has to be examined ex-post.

In the paper we address this ex-post examination. We show that it can be effectively performed for the class of *radial fuzzy systems*. These systems posses a special shape preservation property that enables us to translate the DOC question to a constrained optimization problem. As these optimization problems are generally hard we will show that the optimization can be transferred to the convex case, but for the price of obtaining only a lower bound for DOC.

The rest of the paper is organized as follows. The next section provides the reader with a short introduction into the radial fuzzy systems. Section 3 deals with the covering problem and relates it to the constrained optimization problem. Section 4 concludes the paper.

2 Radial fuzzy systems

We generally consider fuzzy systems in the MISO (multiple-input single-output) configuration and consisting of m IF-THEN rules. Antecedents of rules are therefore represented by multidimensional fuzzy sets $A_j(\boldsymbol{x})$, $j = 1, \ldots, m$ and the system encodes a function from an input space $X \subseteq \mathbb{R}^n$ into an output space $Y \subseteq \mathbb{R}$.

In radial fuzzy systems, we consider that in the i-th dimension, $i = 1, \ldots, n$, $n \in \mathbb{N}$ the one-dimensional fuzzy sets are specified according to formula

$$A(x_i) = act\left(\frac{|x_i - a|}{b}\right) \tag{1}$$

where $a \in \mathbb{R}$ is the central point; $b > 0$ is a scaling parameter and act is a non-increasing shape function such that $act(0) = 1$ and $\lim_{z \to 0} act(z) = 0$. Hence, the membership functions of such a defined one-dimensional fuzzy sets are radial. Notorious examples of these sets are the symmetric triangular fuzzy sets with $act(z) = \max\{1 - z, 0\}$ and Gaussian fuzzy sets with $act(z) = \exp(-z^2)$, see Fig. 1.

In multiple dimensions, a membership function of a radial fuzzy set writes as

$$A(\boldsymbol{x}) = act\,(||\boldsymbol{x} - \boldsymbol{a}||_{\boldsymbol{b}}) \tag{2}$$

where $||\cdot||_{\boldsymbol{b}}$ is a scaled ℓ_p norm with the scaling parameter $\boldsymbol{b} = (b_1, \ldots, b_n)$. These norms are specified for $p \in [1, \infty)$ and $\boldsymbol{u} \in \mathbb{R}^n$ as

$$||\boldsymbol{u}|| = \left(\sum_{i=1}^{n} (|u_i|/b_i)^p\right)^{1/p} \tag{3}$$

and $||\boldsymbol{u}|| = \max\{|u_1|, \ldots, |u_n|\}$ for $p = \infty$.

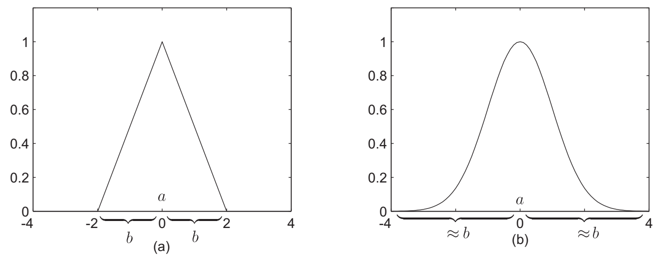

Figure 1: An example of one-dimensional radial fuzzy sets; (a) triangular fuzzy set; (b) Gaussian fuzzy set.

The well-known examples of these norms are the scaled ℓ_1 ($p = 1$) norm, and the Euclidean ($p = 2$) and cubic ($p = \infty$) norms. The latter two norms we will denote by $||\cdot||_{E_{\boldsymbol{b}}}$ and $||\cdot||_{C_{\boldsymbol{b}}}$, respectively. We can easily see that the multivariate version (2) is an extension of the univariate case (1); and (2) represents a radial function as it is well-known in the theory of radial basis neural networks [3].

The *radial property* then requires that the following equality holds when one-dimesional fuzzy sets are combined by a fuzzy conjunction $\star$ (for all rules $j = 1, \ldots, m$):

$$act\left(\frac{|x_1 - a_{j1}|}{b_{j1}}\right) \star \ldots \star act\left(\frac{|x_1 - a_{jn}|}{b_{jn}}\right) = act(||\boldsymbol{x} - \boldsymbol{a}_j||_{\boldsymbol{b}_j}) \tag{4}$$

In (4), we have $\boldsymbol{a}_j = (a_{j1}, \ldots, a_{jn})$ and $\boldsymbol{b}_j = (b_{j1}, \ldots, b_{jn})$; and paraphrasing it in words, it says that the shape (the *act* function) of one-dimensional fuzzy sets is preserved after their combination into a multidimensional fuzzy set by a selected fuzzy conjunction.

Note that a selection of the *act* function and the t-norm $\star$ representing a fuzzy conjunction in (4) is not completely free. For example, when triangular fuzzy sets are combined by a product t-norm one does not obtain a multidimensional triangular fuzzy set. Concerning the question what shapes and t-norms can be combined the reader is referred to [2] where the allowed combinations are thoroughly discussed. However, generally speaking we can say that if the employed t-norm is minimum, then (4) holds for any *act* function with $||\cdot||_{\boldsymbol{b}_j}$ being the scaled norm, i.e., $p = \infty$. In the case of an Archimedean t-norm [4, 5], the *act* function has to correspond to the pseudo-inverse of its additive generator $t^{(-1)}$ in such a way that $act(z) = t^{(-1)}(qz^p)$, for $q > 0$ and $p \in (1, \infty)$, in order to the radial property (4) hold.

2.1 Examples of radial fuzzy systems

Let us present two most important examples of radial fuzzy systems. These are the Mamdani and Gaussian fuzzy systems. An example of antrecedents of these systems in the two-dimensional case is presented in Fig. 2.

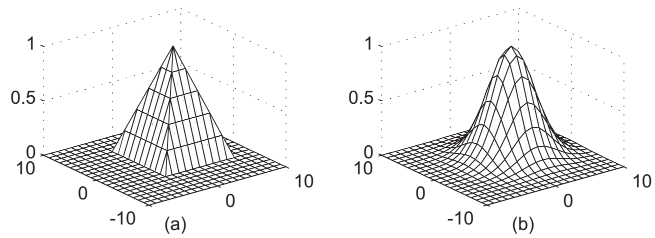

Figure 2: An example of two-dimensional radial fuzzy sets; (a) Mamdani fuzzy system; (b) Gaussian fuzzy system.

2.1.1 Mamdani fuzzy system

In this system, the t-norm is the minimum and the shape function is specified as $act(z) = \max\{0, 1-z\}$ yielding the triangular membership functions. The combination of one-dimensional triangular fuzzy sets yields a multivariate triangular fuzzy set in the form of $A_j(\boldsymbol{x}) = act(\max\{|x_1 - a_{j1}|/b_{j1}, \ldots, |x_n - a_{jn}|/b_{jn}\}) = \max\{0, 1 - ||\boldsymbol{x} - \boldsymbol{a}_j||_{C_{\boldsymbol{b}_j}}\}$, $j = 1, \ldots, m$.

2.1.2 Gaussian fuzzy system

In this system the act function writes as $act(z) = \exp(-z^2)$, the employed t-norm is the product. The additive generator of the product t-norm is $t(z) = -\ln(z)$ and its pseudo-inverse $t^{(-1)}(z) = \exp(-z)$. Thus, setting $q = 1$, $p = 2$ gives the specification of the act function as presented. Clearly, this specification yields the membership functions of fuzzy sets being Gaussian curves and the radial property is related to the well-known fact that a product of Gaussian curves provides again a Gaussian curve. Mathematically, the employed norm is the scaled Euclidean norm so antecedents writes as $A_j(\boldsymbol{x}) = \exp(-||\boldsymbol{x} - \boldsymbol{a}_j||^2_{E_{\boldsymbol{b}_j}})$, $j = 1, \ldots, m$.

3 Degree of covering

The benefits of the radial systems can be identified when we deal with the question of the minimal degree of firing (DOF) across an input space of the system, i.e., with the question of the degree of covering (DOC).

Let the input space of a radial fuzzy system be $X \subseteq \mathbb{R}^n$ and the rule of the system consists of $m \in \mathbb{N}$ rules with antecedents $A_j(\boldsymbol{x}) = A_{j1}(x_1) \star \cdots \star A_{jn}(x_n) = act(||\boldsymbol{x} - \boldsymbol{a}_j||_{\boldsymbol{b}_j})$. We ask what is the minimal degree of firing of the rule base of the system across X, that is we are looking for the value of

$$\mathrm{DOC} = \min_{\boldsymbol{x} \in X}\{\max\{A_1(\boldsymbol{x}), \ldots, A_m(\boldsymbol{x})\}\}.$$

Because *act* is a non-increasing function we have the following chain:

$$\begin{aligned}
\mathrm{DOC} &= \min_{\boldsymbol{x} \in X}\{\max_j\{A_j(\boldsymbol{x})\}\} \\
&= \min_{\boldsymbol{x} \in X}\{\max_j\{act(||\boldsymbol{x}-\boldsymbol{a}_j||_{\boldsymbol{b}_j})\}\} \\
&= \min_{\boldsymbol{x} \in X}\{act(\min_j\{||\boldsymbol{x}-\boldsymbol{a}_j||_{\boldsymbol{b}_j}\})\} \\
&= act(\max_{\boldsymbol{x} \in X}\{\{\min_j\{||\boldsymbol{x}-\boldsymbol{a}_j||_{\boldsymbol{b}_j}\}\}).
\end{aligned}$$

Searching for the maxima of $\min_j\{||\boldsymbol{x}-\boldsymbol{a}_j||_{\boldsymbol{b}_j}\}$ over X is a constrained optimization problem which must be generally solved numerically. On the other hand, due to the radial character of $A_j(\boldsymbol{x})$ we can identify a lower bound for DOC (denoted DOC*) in such a way that it leads to the constrained optimization problem for a convex function [1].

Indeed, $\min_j\{||\boldsymbol{x}-\boldsymbol{a}_j||_{\boldsymbol{b}_j}\} \leq 1/m \sum_j ||\boldsymbol{x}-\boldsymbol{a}_j||_{\boldsymbol{b}_j}$, and therefore one has

$$\mathrm{DOC} \;\geq\; \mathrm{DOC}^* = act\left(\frac{1}{m}\max_{\boldsymbol{x} \in X}\left\{\sum_j ||\boldsymbol{x}-\boldsymbol{a}_j||_{\boldsymbol{b}_j}\right\}\right). \qquad (5)$$

Maximization of function $f(\boldsymbol{x}) = \sum_j ||\boldsymbol{x}-\boldsymbol{a}_j||_{\boldsymbol{b}_j}$, $\boldsymbol{x} \in X$ has to be done again numericaly, but $f(\boldsymbol{x})$ is a convex function so the optimization is practically much easily tractable than in the general case.

To show that $f(\boldsymbol{x})$ is a convex function remind that norms and therefore also scaled ℓ_p norms are convex functions on $\mathbb{R}^n$. A function $h : \mathbb{R}^n \to \mathbb{R}$ is convex if for any two $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^n$, $\alpha \in [0,1]$ and $\beta = 1-\alpha$ one has $h(\alpha\boldsymbol{x}_1 + \beta\boldsymbol{x}_2) \leq \alpha h(\boldsymbol{x}_1) + \beta h(\boldsymbol{x}_2)$. Thus, due to the convexity of ℓ_p norms, we have for any $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^n$ and $\alpha \in [0,1]$, $\beta = 1-\alpha$ the following:

$$||\alpha\boldsymbol{x}_1 + \beta\boldsymbol{x}_2 - \boldsymbol{a}_j||_{\boldsymbol{b}_j} \leq \alpha||\boldsymbol{x}_1 - \boldsymbol{a}_j||_{\boldsymbol{b}_j} + \beta||\boldsymbol{x}_2 - \boldsymbol{a}_j||_{\boldsymbol{b}_j}.$$

Because $f(\alpha\boldsymbol{x}_1 + \beta\boldsymbol{x}_2) = \sum_j ||\alpha\boldsymbol{x}_1 + \beta\boldsymbol{x}_2 - \boldsymbol{a}_j||_{\boldsymbol{b}_j}$ one has

$$\begin{aligned}
f(\alpha\boldsymbol{x}_1 + \beta\boldsymbol{x}_2) &\leq \alpha\sum_j ||\boldsymbol{x}_1 - \boldsymbol{a}_j||_{\boldsymbol{b}_j} + \beta\sum_j ||\boldsymbol{x}_2 - \boldsymbol{a}_j||_{\boldsymbol{b}_j}, \\
f(\alpha\boldsymbol{x}_1 + \beta\boldsymbol{x}_2) &\leq \alpha f(\boldsymbol{x}_1) + \beta f(\boldsymbol{x}_2),
\end{aligned}$$

which is the announced convexity of function $f(\boldsymbol{x})$.

Apparently, $\lim_{\boldsymbol{x}\to\infty} f(\boldsymbol{x}) = \infty$. However, we consider a constrained optimization problem for $\boldsymbol{x} \in X$. Because $f(\boldsymbol{x})$ is convex, every local maxima is also global maxima, and the extreme is reached at the boundary of X if X is compact subset of $\mathbb{R}^n$.

4 Conclusions

In the paper, we have addressed the question of the reliability of knowledge stored in a rule base of a fuzzy system. Quantitatively, this reliability is expressed as the minimal degree of firing (DOF) across the input space of the system. This minimal degree is denoted as the degree of covering (DOC) of the rule base of the system.

We have discussed the specification of the DOC for the case of radial fuzzy systems. As the result, we have shown that it leads to a constrained optimization problem that has to be generally solved numerically. However, due to the specific properties of radial systems, we have shown that a lower bound on DOC can be specified on the basis of maximization of a convex function, which is computationally more tractable than the general case.

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