

On Possibilistic Portfolio Selection Models

R. Fullér¹

¹Óbuda University, Institute of Applied Mathematics
Bécsi út 96/b, H-1034 Budapest, Hungary
E-mail: fuller.robert@nik.uni-obuda.hu

Abstract: We consider optimal portfolio selection problems in a possibilistic setting. Using the possibilistic framework, we can integrate more efficiently the experts' knowledge and the investors' subjective opinions into a portfolio selection model. In 2002 Carlsson, Fullér and Majlender considered portfolio selection problems under trapezoidal possibility distributions and presented an algorithm of complexity $O(n^3)$ for finding an exact optimal solution to the n -asset portfolio selection problem. In this paper we will give a short survey of some works, which extend and develop this possibilistic portfolio selection model.

Keywords: *Fuzzy number, Possibility, Portfolio selection*

1. Introduction

In 1952 Markowitz [19] published his pioneering mean-variance model for optimal portfolio selection problem, which played an important role in the development of modern portfolio selection theory. The basic principle of the mean-variance model is to use the expected return of a portfolio as the investment return and to use the variance of the expected returns of the portfolio as the investment risk. Most of existing portfolio selection models are based on probability theory. However, in many important cases it might be easier to estimate the possibility distributions of rates of return on securities rather than the corresponding probability distributions. Decision makers are usually provided with information which is characterized by linguistic terms for risk, profit and interest rate. Using fuzzy quantities and/or fuzzy constraints they can represent the imperfect knowledge about the returns on the assets and the uncertainty involved in the behaviour of financial markets. We will assume that the rates of return on securities are modelled by possibility distributions rather than probability distributions. Fuzzy numbers can be considered as

possibility distributions [25]. A fuzzy number A is a fuzzy set of the real line with a normal, (fuzzy) convex and upper semi-continuous membership function of bounded support. The family of fuzzy numbers will be denoted by \mathcal{F} . Let A be a fuzzy number. Then $[A]^\gamma$ is a closed convex (compact) subset of \mathbb{R} for all $\gamma \in [0,1]$. Let us introduce the notations $a_1(\gamma) = \min[A]^\gamma$ and $a_2(\gamma) = \max[A]^\gamma$. In other words, $a_1(\gamma)$ denotes the left-hand side and $a_2(\gamma)$ denotes the right-hand side of the γ -cut. A fuzzy set A is called trapezoidal fuzzy number with tolerance interval $[a,b]$, left width α and right width β if its membership function has the following form

$$A(t) = \begin{cases} 1 - \frac{a-t}{\alpha} & \text{if } a - \alpha \leq t \leq a \\ 1 & \text{if } a \leq t \leq b \\ 1 - \frac{t-b}{\beta} & \text{if } a \leq t \leq b + \beta \\ 0 & \text{otherwise} \end{cases}$$

and we will use the notation $A = (a, b, \alpha, \beta)$. The support of A is $(a - \alpha, b + \beta)$. Recall that if $A \in \mathcal{F}$ is a fuzzy number with $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$, $\gamma \in [0,1]$, then the possibilistic mean (or expected) value and variance of A is defined as [2]

$$E(A) = \int_0^1 \gamma(a_1(\gamma) + a_2(\gamma))d\gamma, \quad \sigma^2(A) = \frac{1}{2} \int_0^1 \gamma(a_2(\gamma) - a_1(\gamma))^2 d\gamma.$$

It is easy to see that if $A = (a, b, \alpha, \beta)$ is a trapezoidal fuzzy number then

$$E(A) = \frac{a+b}{2} + \frac{\beta - \alpha}{6}.$$

and

$$\sigma^2(A) = \frac{(b-a)^2}{4} + \frac{(b-a)(\alpha+\beta)}{6} + \frac{(\alpha+\beta)^2}{24} = \left(\frac{b-a}{2} + \frac{\alpha+\beta}{6} \right)^2 + \frac{(\alpha+\beta)^2}{72}.$$

Carlsson, Fullér and Majlender [3] introduced a possibilistic approach for selecting portfolios with the highest utility value under the assumption that the returns of assets are trapezoidal fuzzy numbers. In this paper we give a short survey of some later works that extend and develop their original model.

2. Possibilistic Portfolio Selection Models

We will assume that each investor can assign a utility score to competing investment portfolios based on the expected return and risk of those portfolios. The utility score may

be viewed as a means of ranking portfolios. Portfolios with higher utility values are more attractive. The most commonly employed utility function assigns a risky portfolio P with a risky rate of return r_P , an expected rate of return $E(r_P)$ and a variance of the rate of return $\sigma^2(r_P)$ the following utility score [1]:

$$U(P) = E(r_P) - 0.005 \times A \times \sigma^2(r_P), \quad (1)$$

where A is an index of the investor's risk aversion ($A \approx 2.46$ in US [1]). The factor of 0.005 is a scaling convention that allows us to express the expected return and standard deviation in equation (1) as percentages rather than decimals. Equation (1) is consistent with the notion that utility is enhanced by high expected returns and diminished by high risk. In the mean-variance context, an optimal portfolio selection can be formulated as the following quadratic mathematical programming problem

$$\begin{aligned} U\left(\sum_{i=1}^n r_i x_i\right) &= E\left(\sum_{i=1}^n r_i x_i\right) - 0.005 \times A \times \sigma^2\left(\sum_{i=1}^n r_i x_i\right) \rightarrow \max \\ \text{subject to } &\left\{ \sum_{i=1}^n x_i = 1, x_i \geq 0, i = 1, \dots, n \right\}, \end{aligned}$$

where n is the number of available securities, x_i is the proportion invested in security (or asset) i , and r_i denotes the risky rate of return on security i , $i = 1, \dots, n$.

In 2002 Carlsson, Fullér and Majlender [3] stated the portfolio selection problem with non-interactive possibility distributions as the following quadratic mathematical programming problem

$$\begin{aligned} U\left(\sum_{i=1}^n r_i x_i\right) &= E\left(\sum_{i=1}^n r_i x_i\right) - 0.005 \times A \times \sigma^2\left(\sum_{i=1}^n r_i x_i\right) \rightarrow \max \quad (2) \\ \text{subject to } &\{x_1 + \dots + x_n = 1, x_i \geq 0, i = 1, \dots, n\}. \end{aligned}$$

where $r_i = (a_i, b_i, \alpha_i, \beta_i)$, $i = 1, \dots, n$ are fuzzy numbers of trapezoidal form, and where

$$E\left(\sum_{i=1}^n r_i x_i\right) = \sum_{i=1}^n \frac{1}{2} \left(a_i + b_i + \frac{1}{3}(\beta_i - \alpha_i) \right) x_i,$$

and

$$\sigma^2\left(\sum_{i=1}^n r_i x_i\right) = \left(\sum_{i=1}^n \frac{1}{2} \left(b_i - a_i + \frac{1}{3}(\alpha_i + \beta_i) \right) x_i \right)^2 + \frac{1}{72} \left(\sum_{i=1}^n (\alpha_i + \beta_i) x_i \right)^2.$$

In the literature this model is called as the Carlsson-Fullér-Majlender's Trapezoidal Possibility Model (see [8]).

3. Recent developments

Drawing heavily on [4] in this Section will give a short chronological survey of some later works on possibilistic portfolio selection models. In 2005 Zmeskal [31] introduced an approach to modelling uncertainty of the international index portfolio by the value at risk (VAR) methodology under soft conditions by fuzzy-stochastic methodology. The generalized term uncertainty is understood to have two aspects: risk modelled by probability (stochastic methodology) and vagueness sometimes called impreciseness, ambiguity, softness is modelled by fuzzy methodology. Thus, hybrid model is called fuzzy-stochastic model.

In 2006 Huang [12] considered two types of credibility-based portfolio selection model are provided according to two types of chance criteria. By one chance criterion, the objective is to maximize the investor's return at a given threshold confidence level; by another chance criterion, the objective is to maximize the credibility of achieving a specified return level subject to the constraints. In 2007 Zhang [26] discussed the portfolio selection problem for bounded assets based on the lower and upper possibilistic means and variances of fuzzy numbers. He also pointed out that the lower possibilistic efficient portfolios construct the lower possibilistic efficient frontier. All the upper possibilistic efficient portfolios construct the upper possibilistic efficient frontier. Solving the the resulting problems for all tolerated risk level, the lower and upper possibilistic efficient frontiers are derived explicitly. Smimou et al [22] considered the derivation of portfolio modelling under a fuzzy situation using probability theory, and provides various other (non-probabilistic) scenarios with their utility in risk modelling. They also proposed a simple method for identification of mean-entropic frontier.

In 2008 Gupta et al [9] incorporated fuzzy set theory into a semi-absolute deviation portfolio selection model five criteria: short term return, long term return, dividend, risk and liquidity. In their proposed model, for a given return level, the investor penalizes the negative semi-absolute deviation that is defined as a risk. From computational point of view, the semi-absolute deviation halves the number of required constraints with respect to the absolute deviation. Huang [13] presented two mean-semivariance models for fuzzy model portfolio selection and provided a fuzzy simulation based genetic algorithm to solve these optimization problems. Vercher [23] considered a capital market with n risky assets and a risk-free asset with a fixed rate of return. It is assumed that that each investor can assign a preference level to competing investment portfolios based on the expected return and risk of those portfolios. Then he derived the optimal portfolio using semi-infinite programming in a soft framework. In 2009 Hasuike and Ishii [10] proposed several mathematical models with respect to portfolio selection problems, particularly using the scenario model including the ambiguous factors. These mathematical programming problems with probabilities

and possibilities are called to stochastic programming problem and fuzzy programming problem, and it is difficult to find the global optimal solution for those problems. They also managed to develop an efficient solution method to find the global optimal solution of such a nonlinear programming problem. Zhang et al [27] proposed a new portfolio selection model with the maximum utility based on the interval-valued possibilistic mean and possibilistic variance, which is a two-parameter quadratic programming problem. They also presented a sequential minimal optimization (SMO) algorithm to obtain the optimal portfolio. The remarkable feature of their algorithm is that it is extremely easy to implement, and it can be extended to any size of portfolio selection problems for finding an exact optimal solution. Chen [5] developed two weighted possibilistic portfolio selection models with bounded constraint, which can be transformed to linear programming problems under the assumption that the returns of assets are trapezoidal fuzzy numbers. Dia [7] presented a four-step methodology based on Data Envelopment Analysis for portfolio selection of decision-making units (DMUs) which can be stocks or other financial assets. Along the steps of the methodology, DMUs efficiency ratios are first computed, and then, the generation of a portfolio is carried out by a mathematical model which optimizes the weighted sum of the DMUs' efficiency ratios included in this portfolio, which is optimal for the Decision Maker's system of preferences. Huang [14] showed some credibilistic portfolio selection models, including mean-risk model, mean-variance model, mean-semivariance model, credibility maximization model, α -return maximization model, entropy optimization model and game models.

In 2010 Zhang et al [28] introduced a possibilistic risk tolerance model for the portfolio adjusting problem based on possibility moments theory. An SMO-type decomposition method is developed for finding exact optimal portfolio policy without extra matrix storage. Furthermore, they presented a simple method to estimate the possibility distributions for the returns of assets. Chen [6] considered securities with fuzzy rate of return and developed a possibilistic mean-variance safety-first portfolio model. Using the possibilistic means and variances, he transformed the possibilistic programming problem into a linear optimal problem with an additional quadratic constraint and proposed a cutting plane algorithm to solve it. Petreska and Kolemisevska-Gugulovska [21] introduced a methodology that can be useful in the management of assets against certain given liability and risk estimation of different portfolio structures. In 2011 Huang [15] considered the uncertain portfolio selection problem when security returns cannot be well reflected by historical data. He proposed that uncertain variable should be used to reflect the experts' subjective estimation of security returns. Regarding the security returns as uncertain variables, he introduced a risk curve and developed a mean-risk model. Zhang et al [29] dealt with the portfolio adjusting problem for an existing portfolio under the assumption that the returns of risky assets are fuzzy numbers and there exist transaction costs in portfolio adjusting process. They proposed a portfolio optimization model with V-shaped transaction cost which is

associated with a shift from the current portfolio to an adjusted one.

In 2012 Huang and Qiao [16] discussed a multi-period portfolio selection problem when security returns are given by experts' evaluations. They regarded security return rates as uncertain variables and proposed an uncertain risk index adjustment model. Li et al. [17] proposed an expected regret minimization model, which minimizes the expected value of the distance between the maximum return and the obtained return associated with each portfolio. They proved that their model is advantageous for obtaining distributive investment and reducing investor regret. In 2013 Liu and Zhang [18] discussed a multi-objective portfolio optimization problem for practical portfolio selection in fuzzy environment, in which the return rates and the turnover rates are characterized by fuzzy variables. Based on the possibility theory, fuzzy return and liquidity are quantified by possibilistic mean, and market risk and liquidity risk are measured by lower possibilistic semivariance. Then they propose two possibilistic mean-semivariance models with real constraints. Hasuike and Katagiri [11] considered a robust portfolio selection problem with an uncertainty set of future returns and satisfaction levels in terms of the total return and robustness parameter.

In 2014 Nguyen et al. [20] introduced a new portfolio risk measure that is the uncertainty of portfolio fuzzy return, furthermore they formulate two portfolio optimization models in which the uncertainty of portfolio fuzzy returns is minimized whilst the fuzzy Sharpe ratio is maximized. Zhang et al. [30] considered a multi-period portfolio selection problem imposed by return demand and risk control in a fuzzy investment environment, in which the returns of assets are characterized by fuzzy numbers.

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