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# Flexible fuzzy relation equations

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Abstract: Adjoint triples have been considered in several frameworks, such as in logic programming, formal concept analysis and rough set. Multi-adjoint relation equations are based on these triples and provide a general and flexible setting which cover a large range of applications. This paper presents these equations and several results, which are given thank relationship to concept lattice theory.

*Keywords: Fuzzy relation equations, Galois connection, residuated operators.*

## 1. Introduction

Fuzzy relation equations, introduced by E. Sanchez [14], are associated with the composition of fuzzy relations and have been used to investigate theoretical and applicational aspects of fuzzy set theory [3], e.g. approximate reasoning, time series forecast, decision making, fuzzy control, as an appropriate tool for handling and modeling of non-probabilistic forms of uncertainty, etc. Many papers have investigated the capacity to solve these equations.

Recently, the Galois connection theory has been successfully used to characterize solvability and find a set of solutions of systems of linear-like equations in semilinear spaces [12], which can be interpreted as fuzzy relation equations. In [5], the authors continued with these results and provided a narrow relationship between the solvability of a fuzzy relation equation and the theory of property-oriented concept lattices. These ideas and new results have been extended in a multi-adjoint setting in [4].

This paper presents multi-adjoint relation equations and the interconnection with multi-adjoint concept lattices, specifically, with the property-oriented concept lattices [8].

Several properties of these equations and characterization of both the whole set of solutions and the minimal solutions have been introduced in other papers [6, 11].

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# 2. Multi-adjoint property-oriented concept lattices

The basic operators in this environment are the adjoint triples [1], which are formed by three mappings: a possible non-commutativity conjunctor and two residuated implications, that satisfy the well-known adjoint property.

**Definition 1** *Let* $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ *be posets and* $\&\colon P_1 \times P_2 \to P_3$, $\swarrow\colon P_3 \times P_2 \to P_1$, $\nwarrow\colon P_3 \times P_1 \to P_2$ *be mappings, then* $(\&, \swarrow, \nwarrow)$ *is an* adjoint triple *with respect to* $P_1$,$P_2$,$P_3$ *if:*

$$x \leq_1 z \swarrow y \quad \text{iff} \quad x \,\&\, y \leq_3 z \quad \text{iff} \quad y \leq_2 z \nwarrow x \tag{1}$$

*where* $x \in P_1$, $y \in P_2$ *and* $z \in P_3$.

Equivalence (1) is called *adjoint property*. These operators are a straightforward generalization of t-norms and its residuated implication. Since a t-norm is commutative, in this case both implications coincide. In [10] more general examples of adjoint triples are given.

**Example 2** *Let* $[0,1]_m$ *be a regular partition of* $[0,1]$ *into* $m$ *pieces, for example* $[0,1]_2 = \{0,0.5,1\}$ *divides the unit interval into two pieces.*

*A discretization of a t-norm* $\&\colon [0,1] \times [0,1] \to [0,1]$ *is the operator* $\&^*\colon [0,1]_n \times [0,1]_m \to [0,1]_k$, *where* $n,m,k \in \mathbb{N}$, *and which is defined, for each* $x \in [0,1]_n$ *and* $y \in [0,1]_m$, *as:*

$$x \,\&^*\, y = \frac{\lceil k \cdot (x \,\&\, y) \rceil}{k}$$

*where* $\lceil \_ \rceil$ *is the ceiling function.*

*For this operator, the corresponding residuated implications* $\swarrow^*\colon [0,1]_k \times [0,1]_m \to [0,1]_n$ *and* $\nwarrow_*\colon [0,1]_k \times [0,1]_n \to [0,1]_m$ *are defined as:*

$$z \swarrow^* y = \frac{\lfloor n \cdot (z \leftarrow y) \rfloor}{n} \qquad\qquad z \nwarrow_* x = \frac{\lfloor m \cdot (z \leftarrow x) \rfloor}{m}$$

*where* $\lfloor \_ \rfloor$ *is the floor function and* $\leftarrow$ *is the residuated implication of the t-norm* $\&$.

*The triple* $(\&^*, \swarrow^*, \nwarrow_*)$ *is an adjoint triple, although the operator* $\&^*$ *could be neither commutative nor associative.*

The basic structure, which allows the existence of several adjoint triples for a given triplet of lattices, is the multi-adjoint property-oriented frame.

**Definition 3** *Given two complete lattices* $(L_1, \preceq_1)$ *and* $(L_2, \preceq_2)$*, a poset* $(P, \leq)$ *and adjoint triples with respect to* $P$,$L_2$,$L_1$, $(\&_i, \swarrow^i, \nwarrow_i)$*, for all* $i = 1, \ldots, l$*, a* multi-adjoint property-oriented frame *is the tuple* $(L_1, L_2, P, \&_1, \ldots, \&_l)$.

The definition of context in this framework is analogous to the one given in [9].

**Definition 4** *Let* $(L_1, L_2, P, \&_1, \ldots, \&_l)$ *be a multi-adjoint property-oriented frame. A* context *is a tuple* $(A, B, R, \sigma)$*, where* $A$ *and* $B$ *are non-empty sets (usually interpreted as attributes and objects, respectively),* $R$ *is a* $P$*-fuzzy relation* $R \colon A \times B \to P$ *and* $\sigma \colon B \to \{1, \ldots, l\}$ *is a mapping which associates any element in* $B$ *with some particular adjoint triple in the frame.*

From now on, we will fix a multi-adjoint property-oriented frame and context, $(L_1, L_2, P, \&_1, \ldots, \&_l)$, $(A, B, R, \sigma)$ and, to improve readability, we will write $\&_b$, $\nwarrow_b$ instead of $\&_{\sigma(b)}$, $\nwarrow_{\sigma(b)}$.

In this environment, the following mappings ${}^{\uparrow_\pi} \colon L_2^B \to L_1^A$ and ${}^{\downarrow^N} \colon L_1^A \to L_2^B$ are defined, for each $a \in A$, $b \in B$, as

$$\begin{aligned} g^{\uparrow_\pi}(a) &= \sup\{R(a,b) \&_b g(b) \mid b \in B\} \\ f^{\downarrow^N}(b) &= \inf\{f(a) \nwarrow_b R(a,b) \mid a \in A\} \end{aligned}$$

The pair $({}^{\uparrow_\pi}, {}^{\downarrow^N})$ is an isotone Galois connection [7, 8], that is ${}^{\uparrow_\pi}$ and ${}^{\downarrow^N}$ are order-preserving; and they satisfy that $f^{\downarrow^N \uparrow_\pi} \preceq_1 f$, for all $f \in L_1^A$, and that $g \preceq_2 g^{\uparrow_\pi \downarrow^N}$, for all $g \in L_2^B$.

A pair of fuzzy subsets $\langle g, f \rangle$, with $g \in L^B$, $f \in L^A$, such that $g^{\uparrow_\pi} = f$ and $f^{\downarrow^N} = g$, will be called *multi-adjoint property-oriented concept*. In that case, $g$ is called the *extent* and $f$ the *intent* of the concept. The set of all these concepts will be denoted by $\mathcal{M}_{\pi N}$ and, together with the ordering $\preceq$ defined by $\langle g_1, f_1 \rangle \preceq \langle g_2, f_2 \rangle$ iff $g_1 \preceq_2 g_2$ (or equivalently $f_1 \preceq_1 f_2$), forms a complete lattice [7, 8], $(\mathcal{M}_{\pi N}, \preceq)$, which is called *multi-adjoint property-oriented concept lattice*.

A similar theory is developed if $\sigma$ is defined on $A$, $\sigma \colon A \to \{1, \ldots, l\}$.

# 3. Multi-adjoint relation equations

The multi-adjoint relation equations, given in [4], arise as a generalization of the usual fuzzy relation equations [2, 3, 13], following the philosophy of multi-adjoint framework.

In this section, a multi-adjoint property-oriented frame $(L_1,L_2,P,\&_1,\dots,\&_l)$ will be fixed as the algebraic structure in which the definitions and results will be given.

In general, given the universes $U$, $V$ and $W$, the fuzzy relations $K\colon W \times U \to P$, and $D\colon W \times V \to L_1$, an unknown fuzzy relation $R\colon U \times V \to L_2$, and a mapping that relates each element in $U$ to one adjoint triple, $\sigma\colon U \to \{1,\dots,l\}$, we have that a *multi-adjoint relation equation with* sup-$\&$*-composition* is the equation

$$K \odot_\sigma R = D \tag{2}$$

that is to say, $\bigvee_{u\in U}(K(w,u) \&_u R(u,v)) = D(w,v), \quad w \in W, v \in V$, where $\&_u$ represents the adjoint conjunctor associated with $u$ by $\sigma$, that is, if $\sigma(u) = s$, for $s \in \{1,\dots,l\}$, then $\&_u$ is exactly $\&_s$.

Note that $\sigma\colon U \to \{1,\dots,l\}$ is an interesting mapping, which plays a similar role as the one given in a multi-adjoint context. For instance, this map provides a partition of $U$ in preference sets.

Equation (2) can be rewritten as different systems, one for each $v \in V$, with one equation for each $w \in W$:

$$\bigvee_{u\in U}(K(w,u) \&_u R(u,v)) = D(w,v), \quad w \in W \tag{3}$$

Consequently, if we solve System (3), we attain a column of $R$ (i.e. the elements $R(u,v)$, with $u \in U$). Therefore, solving one system for each $v \in V$, the unknown relation $R$ is obtained. Thus, instead of studying the complete Equation (2) we will consider System (3).

The counterpart of the equation above is a *multi-adjoint relation equation with* inf-$\nwarrow$-*composition*, that is

$$R \lhd_\tau H = E \tag{4}$$

that is to say, $\bigwedge_{v\in V}(R(u,v) \nwarrow_v H(v,w)) = E(u,w), \quad u \in U, w \in W$, where $H\colon V \times W \to P$ and $E\colon U \times W \to L_2$ are fuzzy relations, $R\colon U \times V \to L_1$ is an unknown fuzzy relation.

Therefore, Equation (4) can be written as different systems, one for each $u \in U$:

$$\bigwedge_{v\in V}(R(u,v) \nwarrow_v H(v,w)) = E(u,w), \quad w \in W \tag{5}$$

Note that in System (5), the mapping $\tau$ is defined on $V$ instead of on $U$. This is important in order to relate these equations to a concept lattice framework.

Hence, for each $u \in U$, we obtain a row of $R$ (i.e. the elements $R(u,v)$, with $v \in V$), and so, solving one system for each $u \in U$, the unknown relation $R$ is obtained.

Given the environment needed to define System (3) (resp. System (5)) and *its associated multi-adjoint property-oriented context* $(W,U,K,\sigma)$ *(resp.* $(V,W,H,\tau)$*)*, the concept lattice associated with this context will be called $\mathcal{M}_{\Pi N}(K)$ (resp. $\mathcal{M}^*_{\Pi N}(H)$).

As a consequence of this relation, the following result holds [4].

**Theorem 5** *[4] Let $v \in V$ and the fuzzy subset $f_v \in L_1^W$, defined as $f_v(w) = D(w,v)$, for all $w \in W$. System (3) can be solved if and only if $\langle f_v^{\downarrow^N}, f_v\rangle$ is a concept of $\mathcal{M}_{\Pi N}(K)$. In this case, $g \in L_2^U$ is a solution of System (3) if and only if $g^{\uparrow_\Pi} = f_v$ and $f_v^{\downarrow^N}$ is the greatest solution.*

*Analogously, let $u \in U$ and $g_u \in L_2^W$, defined as $g_u(w) = E(u,w)$, for all $w \in W$. System (5) can be solved if and only if $\langle g_u, g_u^{\uparrow_\Pi}\rangle$ is a concept of $\mathcal{M}^*_{\Pi N}(H)$. In this case, $f \in L_1^V$ is a solution of System (5) if and only if $f^{\downarrow^N} = g_u$ and $g_u^{\uparrow_\Pi}$ is the least solution.*

Note that we are identifying the columns of a matrix $R$ by a fuzzy subset $g \in L_2^U$ and the rows of $R$ by a fuzzy subset $f \in L_1^V$.

More properties were given in [4] and these equations have been studied in other papers.

One important goal in fuzzy relation equation theory is to find out minimal solutions, which is more difficult. In [6, 11] new properties and characterizations of the minimal solutions are presented and so the whole set of solutions can be determined.

# 4. Conclusions and future work

Multi-adjoint relation equations have been introduced, which generalize several extensions of fuzzy relation equations. In this general environment, different conjunctors and residuated implications can be used, which provide more flexibility in order to relate the variables considered in the system and to solve it.

A number of open questions remain for further study, one of them being the complexity of the procedure in the last papers. It is well-known that implications in MV-algebras are infinitely distributive. A topic of future study is to characterize all structures where implication is infinitely distributivity.

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## References


[1] M. Cornejo, J. Medina, and E. Ramírez. A comparative study of adjoint triples. *Fuzzy Sets and Systems*, 211:1–14, 2013.

[2] B. De Baets. Analytical solution methods for fuzzy relation equations. In D. Dubois and H. Prade, editors, *The Handbooks of Fuzzy Sets Series*, volume 1, pages 291–340. Kluwer, Dordrecht, 1999.

[3] A. Di Nola, E. Sanchez, W. Pedrycz, and S. Sessa. *Fuzzy Relation Equations and Their Applications to Knowledge Engineering*. Kluwer Academic Publishers, Norwell, MA, USA, 1989.

[4] J. C. Díaz and J. Medina. Multi-adjoint relation equations: Definition, properties and solutions using concept lattices. *Information Sciences*, 253:100–109, 2013.

[5] J. C. Díaz and J. Medina. Solving systems of fuzzy relation equations by fuzzy property-oriented concepts. *Information Sciences*, 222:405–412, 2013.

[6] J. C. Díaz-Moreno and J. Medina. Using concept lattice theory to obtain the set of solutions of multi-adjoint relation equations. *Information Sciences*, 266(0):218–225, 2014.

[7] J. Medina. Towards multi-adjoint property-oriented concept lattices. *Lecture Notes in Artificial Intelligence*, 6401:159–166, 2010.

[8] J. Medina. Multi-adjoint property-oriented and object-oriented concept lattices. *Information Sciences*, 190:95–106, 2012.

[9] J. Medina, M. Ojeda-Aciego, and J. Ruiz-Calviño. Formal concept analysis via multi-adjoint concept lattices. *Fuzzy Sets and Systems*, 160(2):130–144, 2009.

[10] J. Medina, M. Ojeda-Aciego, A. Valverde, and P. Vojtáš. Towards biresiduated multi-adjoint logic programming. *Lecture Notes in Artificial Intelligence*, 3040:608–617, 2004.

[11] J. Medina, E. Turunen, E. Bartl, and J. C. Díaz-Moreno. Minimal solutions of fuzzy relation equations with general operators on the unit interval. In A. Laurent, O. Strauss, B. Bouchon-Meunier, and R. R. Yager, editors, *IPMU (3)*, volume 444 of *Communications in Computer and Information Science*, pages 81–90. Springer, 2014.

[12] I. Perfilieva. Fuzzy relation equations in semilinear spaces. *Communications in Computer and Information Science*, 80:545–552, 2010.

[13] I. Perfilieva and L. Nosková. System of fuzzy relation equations with inf-$\longrightarrow$ composition: Complete set of solutions. *Fuzzy Sets and Systems*, 159(17):2256–2271, 2008.

[14] E. Sanchez. Resolution of composite fuzzy relation equations. *Information and Control*, 30(1):38–48, 1976.